UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **MATH7202**

MODULE NAME : **Algebra 4: Groups and Rings**

DATE : **01-May-07**

TIME : **14:30**

TIME ALLOWED : **2 Hours 0 Minutes**

2006/07-MATH7202A-001-EXAM-138

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Define the notion of an *automorphism* of a group K.

   Explain how the set $\mathrm{Aut}(K)$ of automorphisms of K can be regarded as group, and describe in detail

   (i) the group structure on $\mathrm{Aut}(C_{10})$;

   (ii) all possible homomorphisms $h : C_4 \to \mathrm{Aut}(C_{10})$.

   Let $h : Q \to \mathrm{Aut}(K)$ be a group homomorphism ; explain carefully how to form *semi-direct product* group

$$K \rtimes_h Q .$$

   Explain with proof, giving generators and relations, how many isomorphically distinct groups there are of the form $C_{10} \rtimes_h C_4$.

2. Let G be a finite group. Explain what is meant by the order, $\mathrm{ord}(g)$, of $g \in G$.

   If x is a generator of the cyclic group C_n of order n, show that, for $1 \leqslant m \leqslant n-1$,

$$\mathrm{ord}(x^m) = \frac{n}{\mathrm{HCF}(m,n)}.$$

   Let $\varphi_m : C_n \to C_n$ denote the homomorphism $\varphi_m(x^t) = x^{mt}$. Derive a necessary and sufficient condition on m for φ_m to be an automorphism.

   Define the *Euler totient function* $\Phi(n)$, and prove that if $p_1, \ldots, p_m$ are the distinct primes dividing the positive integer n then

$$\Phi(n) = n\left(1 - \frac{1}{p_1}\right) \ldots \left(1 - \frac{1}{p_m}\right).$$

   Hence or otherwise, find the order of $\mathrm{Aut}(C_{23000})$.

3. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X. Explain what is meant by

   i) the orbit $\langle x \rangle$ of $x \in X$;

   ii) the stability subgroup G_x of $x \in X$;

   iii) the quotient set G/G_x.

   Show that, for any $x \in X$ there exists a bijection $G/G_x \longleftrightarrow \langle x \rangle$.

   Explain what is meant by the *class equation* of such an action (in each of its forms).

   Furthermore describe the class equation explicitly in the case where $X = G = D_{14}$, the dihedral group of order 14, and the action is *conjugation*

$$\circ \; : \; D_{14} \times D_{14} \to D_{14} \; ; \; g \circ h = ghg^{-1}.$$

4. Let p be a prime, and let G be a group of order p^n ($n \geqslant 1$) acting on a finite set X. Define the *fixed point set* X^G, and prove that

$$|X| \equiv |X^G| \pmod p.$$

   For any integer $k \geqslant 1$, by means of a suitable action of G show that

$$\binom{kp^n}{p^n} \equiv k \pmod p.$$

   Let P, Q be subgroups of a group G; explain what is meant by saying that P *normalizes* Q. Show that, when P normalizes Q, there is a group isomorphism

$$PQ/Q \cong P/(P \cap Q).$$

   Let p be a prime, and let G be a group of order kp^n where $n \geqslant 1$ and k is coprime to p, and let N_p be the number of subgroups of G of order p^n. Assuming that $N_p \geqslant 1$, show that

$$N_p \equiv 1 \pmod p.$$

5. Let $\mathbf{F}$ be a field and let G be a finite subgroup of the multiplicative group $\mathbf{F}^*$. If p is prime and G has order p^n prove that G is cyclic.

   Show that $p(x) = x^3 + x^2 + 1$ is irreducible over $\mathbf{F}_2$, the field with two elements.

   Explain why one may represent elements of $\mathbf{F}_2[x]/(x^3 + x^2 + 1)$ by polynomials of degree $\leqslant 2$.

   Write out the multiplication table of $(\mathbf{F}_2[x]/(x^3 + x^2 + 1))^*$ and describe explicitly an isomorphism $(\mathbf{F}_2[x]/(x^3 + x^2 + 1))^* \cong C_n$, stating in advance the value of n.

6. State and prove Eisenstein's criterion for irreducibility.

   In each case below, decide whether or not the given polynomial is irreducible over $\mathbf{Q}$, jusifying your answer in each case. If the polynomial is not irreducible, give its complete factorization into $\mathbf{Q}$-irreducible factors.

   (i) $x^4 + 4x^3 + 16x^2 + 24x + 13$;

   (ii) $x^8 + 1$.

   (iii) $x^{12} + 1$.